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Some Tree-Star Ramsey Numbers

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The Ramsey Number $r(G_1, G_2)$ is the least integer N such that for every graph G with N vertices, either G has the graph G_1 as a subgraph or $\bar{G}$, the complement of G, has the graph G_2 as a subgraph.

In this paper we embed the paths P_m in a much larger class $\mathscr{T}$ of trees and then show how some evaluations by T. D. Parsons of Ramsey numbers $r(P_m, K_{1,n})$, where $K_{1,n}$ is the star of degree n, are also valid for $r(T_m, K_{1,n})$ where $T_m \in \mathscr{T}$.

1. INTRODUCTION

The quantity $r(G_1, G_2)$ is the least integer N such that for every graph G with N vertices, either G has the graph G_1 as a subgraph or $\bar{G}$, the complement of G, has the graph G_2 as a subgraph.

The well-known theorem of F. Ramsey [4] ensures the existence of the numbers $r(G_1, G_2)$, and many authors are currently evaluating these "Ramsey numbers" for special G_1, G_2. The reader is referred to the excellent survey article by S. Burr [1] for an extensive bibliography.

T. D. Parsons [3] has recently evaluated $r(P_m, K_{1,n})$ where P_m is a path with m vertices and $K_{1,n}$ is the star of degree n. In Section 2 of this note we embed the paths in a large class $\mathscr{T}$ of trees and then in Section 3 show that many of Parsons' results are also true if P_m is replaced by T_m where T_m is any m-vertex member of $\mathscr{T}$. Most of the proofs in Section 3 are identical to those of Parsons and therefore, we do not, repeat them.

Any undefined term or notation is given in Harary's book [2].

2. THE CLASS OF TREES $\mathscr{T}$

LEMMA 1. *If a graph G has $\delta(G) \geqslant k$, then G contains as a subgraph any tree on $k+1$ vertices.*

Proof. By induction on k. The result is true for $k = 1$ and 2. Now consider any graph G with $\delta(G) \geqslant k+1$ and any tree T having $k+2$ vertices. Delete from T an edge $[v, w]$ and vertex w which has degree 1 in T, thus forming a tree T' with $k + 1$ vertices. Delete from G any vertex and all edges adjacent to this vertex, forming G' with $\delta(G') \geqslant k$. By the induction hypothesis, $G' \supset T'$. If $d(x, F)$ denotes the degree of vertex x in the graph F, then $d(v, T') \leqslant k$ and $d(v, G) \geqslant k + 1$. Hence v is adjacent to a vertex of G not in T' and the addition of such an edge produces $T \subset G$, thus completing the proof.

Let T be a spanning tree of G which has m vertices. A permutation π of $V(G)$ is *T-preserving* if and only if for each $[\alpha, \beta] \in E(T)$, $[\pi(\alpha), \pi(\beta)] \in E(G)$. Let x be a vertex of G and X be the set of all images of x under T-preserving permutations of $V(G)$.

We say that (T, x) is a *complete pair* if and only if the assumption that each $x \in X$ has degree $m - 1$ in G implies that G is the complete graph K_m.

As an illustration, consider the pair $(T, 1)$ of Fig. 1:

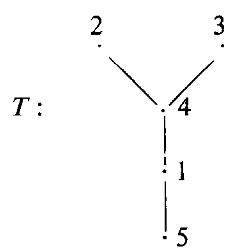

FIGURE 1.

Since $1 \in X$, $d(1, G) = 4$. Therefore

$1 \to 3$	$1 \to 2$	$1 \to 4$
$2 \to 2$	$2 \to 3$	$2 \to 2$
$3 \to 5$	$3 \to 5$	$3 \to 5$
$4 \to 1$	$4 \to 1$	$4 \to 1$
$5 \to 4$	$5 \to 4$	$5 \to 3$

are T-preserving permutations. Hence the vertices 2, 3, 4 are in X and have degree 4 in G. Hence $G = K_5$ and $(T, 1)$ is a complete pair.

The trees P_{2n+1} and $K_{1,n}$ together with their respective center vertices do not form complete pairs. Further, the tree with vertex set

$$\{x_1\} \cup \{x_2\} \cup Y_1 \cup Y_2, \qquad \text{where} \quad |Y_1|, |Y_2| \geqslant 2,$$

and edge set $\{[x_1, x_2]\} \cup \{[x_1, y]: y \in Y_1\} \cup \{[x_2, y]: y \in Y_2\}$ does not form a complete pair with either of the vertices x_1, x_2. We have not as yet completely characterized complete pairs. The following result shows that degree-one vertices always yield complete pairs.

LEMMA 2. *If x has degree one in a spanning tree T of G, then (T, x) is a complete pair.*

Proof. Suppose $|V(G)| = m$. Let the length of the path from vertex v to x be $D(v)$. It is sufficient to prove that for all k, each vertex v with $D(v) = k$ is an image of x under a T-preserving permutation. We do this by induction on k. The case $k = 0$ is trivial. Now assume the result for all $k \leqslant j$ and suppose that $D(w) = j + 1$. Let the vertices in the path from x to w be $x, x_1, \ldots, x_j, w$. By the induction hypothesis, $x, x_1, \ldots, x_j$ have degree $m - 1$ in G. Therefore the permutation which maps the ordered set $\{x, x_1, \ldots, x_j, w\}$ into $\{w, x, x_1, \ldots, x_j\}$ and leaves all other vertices invariant is T-preserving and the image of x is w as required. This completes the proof.

A tree T is in the *class* $\mathscr{T}$ if and only if some vertex v of degree one in T and its incident edge $[v, w]$ may be deleted forming a tree T' where (T', w) is a complete pair.

Lemma 2 implies that $\mathscr{T}$ contains all trees which have a vertex of degree one adjacent to vertex of degree two, hence in particular $\mathscr{T}$ contains all paths P_m. However, there are many trees in $\mathscr{T}$ without this property. No star $K_{1,n}$ for $n \geqslant 3$ is in $\mathscr{T}$. For $m \leqslant 7$, the only tree not in $\mathscr{T}$ which is not a star is that shown in Fig. 2.

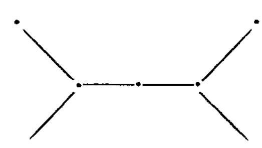

FIG. 2. A tree not in $\mathscr{T}$.

A quick glance at the tree diagrams of [2, p. 233] shows there are at most five trees not in $\mathscr{T}$ among the 23 8-vertex trees.

3. RAMSEY NUMBERS

The following two results have recently appeared in the literature. Let T_m be any tree with m vertices.

THEOREM 1 (Burr [1]).

$$r(T_m, K_{1,n}) \leqslant m + n - 1.$$

THEOREM 2 (Burr [1]). *If* $m-1 \mid n-1$, *then*

$$r(T_m, K_{1,n}) = m+n-1.$$

The next result generalizes Lemma 4 in Parsons' work [3].

LEMMA 3. *If* $T_m \in \mathcal{T}$ *and* $n \not\equiv 1 \pmod{m-1}$, *then*

$$r(T_m, K_{1,n}) \leqslant m+n-2.$$

Proof. By induction on n. The statement is true for $n=1$. Suppose $n>1$, $n \not\equiv 1 \pmod{m-1}$, and G has $m+n-2$ vertices. Suppose $G \not\supset T_m$ and $\Delta(\bar{G}) \leqslant n-1$. Then

$$\delta(G) = m+n-3-\Delta(\bar{G}) \geqslant m+n-3-(n-1) = m-2,$$

and so by definition of $\mathcal{T}$ and Lemma 1, G contains a tree T'_{m-1} formed from T_m by removal of a vertex v of degree one in T_m and its incident edge $[v, w]$, and such that (T'_{m-1}, w) is a complete pair. Let the vertex set of T'_{m-1} be $W = \{v_1, \ldots, v_{m-1}\}$ and the remaining vertices be $H = \{v_m, \ldots, v_{m+n-2}\}$. If the image of w under any T'_{m-1}-preserving permutation of W is adjacent in G to any vertex of H, $G \supset T_m$. Hence there is no such adjacency, and each such image, having degree $\geqslant m-2$ in G, has degree $m-2$ in $\langle W \rangle$, the subgraph induced by W in G. Since (T'_{m-1}, w) is a complete pair, $G = K_{m-1} \cup \langle H \rangle$. If $n \leqslant m-1$, then $d(v_m, \bar{G}) \geqslant m-1 \geqslant n$, hence $n > m-1$. By the induction hypothesis, since $n-m+1 \not\equiv 1 \pmod{m-1}$, $r(T_m, K_{1,n-m+1}) \leqslant m+(n-m+1)-2 = n-1$. But H has $n-1$ vertices and $H \not\supset T_m$ so $\Delta(\bar{H}) \geqslant n-m+1$. Then

$$\Delta(\bar{G}) \geqslant \Delta(\bar{H}) + (m-1) \geqslant n-m+1+m-1 = n,$$

a contradiction. The lemma follows.

LEMMA 4. *If* F_m *is any connected graph with* m *vertices and* $n \equiv 0$, $2 \pmod{m-1}$, *then*

$$r(F_m, K_{1,n}) > m+n-3.$$

Proof. Identical to that of Theorem 4 in [3].

THEOREM 3. *If* $T_m \in \mathcal{T}$ *and* $m \equiv 0, 2 \pmod{m-1}$, *then*

$$r(T_m, K_{1,n}) = m+n-2.$$

Proof. Immediate from Lemmas 3 and 4.

THEOREM 4. *If* $T_m \in \mathscr{T}$, $n \not\equiv 1 \pmod{m-1}$, *and* $n \geqslant (m-3)^2$, *then*

$$r(T_m, K_{1,n}) = m + n - 2.$$

Proof. The same arguments used in the proofs of Lemma 5 and Theorem 5 in [3] establish that for any connected graph F_m with m vertices, where m satisfies the hypotheses, $r(F_m, K_{1,n}) > m + n - 3$, and the result follows from Lemma 3.

THEOREM 5. *If* $T_m \in \mathscr{T}$, $n \not\equiv 1 \pmod{m-1}$, *and* $n \equiv 1 \pmod{m-2}$, *then*

$$r(T_m, K_{1,n}) = m + n - 2.$$

Proof. Let T'_{m-1} be any tree formed from T_m by deleting a vertex of degree one. Then using Theorem 2,

$$r(T_m, K_{1,n}) \geqslant r(T'_{m-1}, K_{1,n}) = (m-1) + n - 1.$$

The result now follows from Lemma 3.

THEOREM 6. *If* $T_m \in \mathscr{T}$, $m > 3$, $n \equiv -1 \pmod{m-1}$, *and* $n \geqslant m - 1$, *then*

$$r(T_m, K_{1,n}) = m + n - 2.$$

Proof. Identical to that of Theorem 7 in [3].

ACKNOWLEDGMENT

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